

Bogoliubov approach to the quantum phase transition of a disordered Bose Hubbard Model in weakly interacting regime

Botao Wang¹ and Ying Jiang^{1,2,*}

¹*Department of Physics, Shanghai University, Shanghai 200444, P.R. China*

²*Qian Weichang College, Shanghai University, Shanghai 200444, P.R. China*

With the help of Bogoliubov theory, we investigate the quantum phase transition of a two-dimensional disordered Bose-Hubbard model in weakly interacting regime. Within the framework of Bogoliubov theory, an analytical expression for the particle density is derived and the dependence of condensate fraction on disorder strength as well as on lattice depth are discussed. Based on the above mentioned dependency relations, the quantum phase boundary between superfluid phase and Bose glass phase in the disorder Bose-Hubbard system in weakly interacting regime is determined. The obtained phase diagram agrees qualitatively with the empirical square-root law, our result may serve as a reference object for possible experimental investigation.

PACS numbers: 67.85.-d, 64.70.Tg, 03.75.Lm

I. INTRODUCTION

The competition between disorder and interaction, which gives rise to various novel physical effects and phenomena, has been an important issue in condensed matter physics. Due to the intrinsic complexity of condensed matter systems as well as the poor control of disorder in experiments, it is very hard to have a thorough investigation of these systems in both theoretical and experimental ways. However, the appearance of ultracold quantum gases in optical lattices opens a new chapter in mimicking the condensed matter systems [1]. Because of the unprecedented level of control and precision, such a so-called quantum simulator turns out to be an ideal tool in studying disordered systems [2, 3].

Nowadays, it is routinely possible to create systems of ultracold atoms in different optical lattices [1–4] since the remarkable experimental work by Greiner *et al.* [5] in which the quantum phase transition from the superfluid (SF) to Mott insulator (MI) was first observed. When disorder is introduced, there emerges a new phase, i.e. Bose glass (BG, quasi-superfluid puddles embedded in an MI insulating background) [6]. Even though disorder, or pseudo-disorder can be realized in optical lattices by means of speckle patterns [7–9], multichromatic incommensurate optical lattices [10], or localized atomic impurities [11], directly observing such BG phase in disordered lattice systems has been a big challenge [10–13]. Only recently was it reported that the phase transition of SF-BG could be detected in 3D optical lattices by measuring the amount of excitations generated by quench [14], while the probe of SF-BG in 2D disordered optical lattices (to our knowledge) is still on the way.

Theoretically, a system with ultracold bosons trapped in optical lattices is often described by Bose Hubbard model (BHM) with the disorder encrypted in the on-

site potential [6, 15]. Fisher *et al.* predicted that it is unlikely, although not rigorously impossible, that there exists a direct transition between SF and MI states for finite disorder [6]. Since then it has been a subject of debate for a long time. A variety of sophisticated methods, both analytical and numerical, have been applied in this study, including density-matrix renormalization group (DMRG) techniques, local mean-field (LMF) approximation, stochastic mean field (SMF) theory, strong coupling expansion, quantum Monte Carlo (QMC) methods as well as local mean-field cluster analysis (for references see Ref.[16] and references therein). Several years ago Pollet *et al.*, by proving the theorem of inclusions, have resolved such a controversy and reported that a direct transition from SF to MI is impossible in disordered lattice system [17]. By means of QMC simulations the phase diagrams of disordered BHM in both 3D [18] and 2D [19, 20] optical lattices have also been given. However, most of the efforts have been only devoted to the phase transition problem in strongly interacting regime, and little attention has been paid to a disordered Bose gas in lattice systems with weak interaction, where a phase transition from SF to BG may also occur when disorder strength is large enough [16–19]. Not only numerical methods fail to give the phase boundary of SF-BG in weakly interacting regime because of the finite size effect [19, 20], but a rigorous analytic investigation of the phase diagram in such a regime is still missing so far, except for an empirical square-root law estimation [19] that has been proposed in some way analogous to the disordered continuum (without optical lattices) case [21]. Thus, it is highly desirable to have a further exploration in the SF-BG phase transition problem of disordered weakly interacting Bose gases in optical lattices.

As more or less a standard mean-field method in weakly interacting systems, Bogoliubov theory (BT) has been extensively used for Bose gases both in absence [22–28] and in presence of optical lattices [29–35]. In continuum cases (without lattices), a dilute Bose gas with weak disorder can be correctly described within the framework

*Corresponding author; Electronic address: yjiang@shu.edu.cn

of BT [24, 25]. As for lattice systems without disorder, it turns out that BT can provide a semi-quantitative description for recent experiments, including measurements of condensate population [36] as well as excitation spectrum [37, 38]. Unfortunately, BT fails to determine the phase boundary of SF-MI in clean lattice systems [31]. This failure is understandable considering MI only appears in strongly interacting regime, while BT is based on the assumption of weak interaction. Whereas, in disordered lattice systems, the disorder-driven SF-BG transition can happen in weak interaction regime [6, 16, 18, 19]. It thus becomes interesting to investigate such a SF-BG phase boundary with the help of BT.

The structure of the paper is following. After introducing the disordered Bose-Hubbard Hamiltonian, we perform the Bogoliubov transformation in Section II. The analytic expression of the particle density is given in Section III. In Section IV, we investigate the phase diagram by evaluating the depletions of the condensate fraction due to the the strength of disorder as well as the depth of optical potential. Finally, we summarize our results in section V.

II. THE MODEL AND BOGOLIUBOV THEORY

An ultracold dilute Bose gas in a 2D disordered optical lattice can be depicted by the following disordered Bose-Hubbard Hamiltonian [6, 15, 16]

$$\hat{H}_{BH} = -t \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \frac{U}{2} \sum_i \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i - \sum_i (\mu + \delta_i) \hat{a}_i^\dagger \hat{a}_i \quad (1)$$

where \hat{a}_i and \hat{a}_i^\dagger are bosonic annihilation and creation operators at site i , satisfying the canonical commutation. t is nearest-neighbor hopping parameter, U represents the on-site repulsive interaction strength and μ is the chemical potential. δ_i describes the random potential, which is assumed to be uniformly distributed between $[-\Delta, \Delta]$ and spatially uncorrelated

$$\overline{\delta_i} = 0, \quad \overline{\delta_i \delta_{i'}} = \frac{\Delta^2}{3} \delta_{i,i'} \quad (2)$$

where $\overline{\cdots}$ stands for the disorder ensemble average and Δ denotes the disorder strength.

After performing Fourier transformations for the operators $\hat{a}_i = (1/N_S)^{1/2} \sum_{\mathbf{k}} \hat{a}_{\mathbf{k}} e^{-i \cdot \mathbf{k}}$ (also for its corresponding conjugate) and the random potential $\delta_i = (1/N_S)^{1/2} \sum_{\mathbf{k}} \delta_{\mathbf{k}} e^{-i \cdot \mathbf{k}}$, the Hamiltonian Eq. (1) can be

rewritten in momentum space as

$$\begin{aligned} \hat{H} = & - \sum_{\mathbf{k}} \left(2t \sum_{l=1}^d \cos a k_l + \mu \right) \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} \\ & + \frac{U}{2} \frac{1}{N_S} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4} \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2}^\dagger \hat{a}_{\mathbf{k}_3} \hat{a}_{\mathbf{k}_4} \delta_{\mathbf{k}_1 + \mathbf{k}_2, \mathbf{k}_3 + \mathbf{k}_4} \\ & - \frac{1}{\sqrt{N_S}} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3} \delta_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2}^\dagger \hat{a}_{\mathbf{k}_3} \delta_{\mathbf{k}_2, \mathbf{k}_1 + \mathbf{k}_3}, \end{aligned} \quad (3)$$

N_S is the number of lattice sites, a is the lattice constant, d denotes the dimension and \mathbf{i} represents the coordinate of site i . Near absolute zero temperature, the number of atoms in the state with $\mathbf{k} = \mathbf{0}$ becomes macroscopically large, which allows the so called Bogoliubov approximation $\hat{a}_{\mathbf{0}} \simeq \hat{a}_{\mathbf{0}}^\dagger \simeq \sqrt{N_0}$ [22]. In such a case, retaining all terms up to second order in $\hat{a}_{\mathbf{k}}^\dagger$, $\hat{a}_{\mathbf{k}}$ and $\delta_{\mathbf{k}}$ yields

$$\begin{aligned} \hat{H} = & \left(\frac{U}{2} n_0 - 2dt - \mu \right) N_0 + \sum_{\mathbf{k} \neq \mathbf{0}} A_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} \\ & + \frac{U}{2} n_0 \sum_{\mathbf{k} \neq \mathbf{0}} \left(\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger + \hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}} \right) \\ & - \sqrt{n_0} \sum_{\mathbf{k} \neq \mathbf{0}} \left(\delta_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger + \delta_{-\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger \right), \end{aligned} \quad (4)$$

where we have defined the condensate density $n_0 = N_0/N_S$ and introduced for brevity

$$A_{\mathbf{k}} = 2U n_0 - \mu - 2t \sum_{l=1}^d \cos a k_l. \quad (5)$$

The above Hamiltonian can be diagonalized by the inhomogeneous Bogoliubov transformation [23, 26]

$$\begin{cases} \hat{a}_{\mathbf{k}}^\dagger = u_{\mathbf{k}} \hat{b}_{\mathbf{k}}^\dagger - v_{\mathbf{k}} \hat{b}_{-\mathbf{k}} - z_{\mathbf{k}} \\ \hat{a}_{\mathbf{k}} = u_{\mathbf{k}} \hat{b}_{\mathbf{k}} - v_{\mathbf{k}} \hat{b}_{-\mathbf{k}}^\dagger - z_{\mathbf{k}}. \end{cases} \quad (6)$$

This transformation to new operators $\hat{b}_{\mathbf{k}}^\dagger$ ($\hat{b}_{\mathbf{k}}$) should preserve canonical commutation relations, which gives $u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2 = 1$. A calculation along the standard procedure leads to the following Bogoliubov parameters

$$v_{\mathbf{k}}^2 = \frac{1}{2} \left(\frac{A_{\mathbf{k}}}{E_{\mathbf{k}}} - 1 \right), \quad u_{\mathbf{k}}^2 = \frac{1}{2} \left(\frac{A_{\mathbf{k}}}{E_{\mathbf{k}}} + 1 \right), \quad (7)$$

$$z_{\mathbf{k}} = - \frac{\sqrt{n_0}}{A_{\mathbf{k}} + U n_0} \delta_{\mathbf{k}}, \quad (8)$$

and the Bogoliubov spectrum

$$E_{\mathbf{k}} = \sqrt{A_{\mathbf{k}}^2 - (U n_0)^2}. \quad (9)$$

In the end, the diagonalized Hamiltonian reads

$$\begin{aligned} \hat{H} = & \left(\frac{U n_0}{2} - 2dt - \mu \right) N_0 + \frac{1}{2} \sum_{\mathbf{k} \neq \mathbf{0}} (E_{\mathbf{k}} - A_{\mathbf{k}}) \\ & - \sum_{\mathbf{k} \neq \mathbf{0}} \frac{n_0}{A_{\mathbf{k}} + U n_0} \delta_{\mathbf{k}}^2 + \sum_{\mathbf{k} \neq \mathbf{0}} E_{\mathbf{k}} \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}}. \end{aligned} \quad (10)$$

III. PARTICLE DENSITY

With the diagonalized Hamiltonian (10) in hand, the corresponding grand canonical potential Ω can be obtained straightforwardly by means of

$$\Omega = -\frac{1}{\beta} \ln \mathcal{Z}, \quad \mathcal{Z} = \text{Tre}^{-\beta \hat{H}} \quad (11)$$

where \mathcal{Z} is the grand canonical partition function and $\beta = 1/(k_B T)$ is the reciprocal temperature with k_B being Boltzman constant. Note that since the grand canonical potential changes with each realization of the disorder potential, we obtain the averaged thermodynamic potential after performing the disorder ensemble average

$$\begin{aligned} \bar{\Omega} = & \left(\frac{Un_0}{2} - \mu - 2dt \right) n_0 N_s + \frac{1}{2} \sum_{\mathbf{k} \neq 0} (E_{\mathbf{k}} - A_{\mathbf{k}}) \\ & - \frac{\Delta^2}{3} \sum_{\mathbf{k} \neq 0} \frac{n_0}{A_{\mathbf{k}} + Un_0} - \frac{1}{\beta} \sum_{\mathbf{k} \neq 0} \ln(1 - e^{-\beta E_{\mathbf{k}}}) \end{aligned} \quad (12)$$

where $\bar{\delta}_{\mathbf{k}} = 0$, $\bar{\delta}_{\mathbf{k}}^2 = \Delta^2/3$. The above expression indicates that the grand canonical potential contains mean-field result (the first term on the right of the above expression), contributions from quantum fluctuations (the middle two terms) and thermal fluctuation (the last term).

With the help of the thermodynamic relation $n = -(1/N_S) \partial \bar{\Omega} / \partial \mu$, we obtain a general expression of total particle density in the framework of Bogoliubov theory

$$n = n_0 + n_I + n_T + n_R \quad (13)$$

which contains the condensate density n_0 , the condensate depletion due to the on-site repulsive interaction

$$n_I = \frac{1}{2} \frac{1}{N_S} \sum_{\mathbf{k} \neq 0} \left(\frac{A_{\mathbf{k}}}{E_{\mathbf{k}}} - 1 \right), \quad (14)$$

the temperature induced depletion

$$n_T = \frac{1}{N_S} \sum_{\mathbf{k} \neq 0} \frac{1}{e^{\beta E_{\mathbf{k}}} - 1} \frac{A_{\mathbf{k}}}{E_{\mathbf{k}}}, \quad (15)$$

as well as the condensate depletion coming from the random potential

$$n_R = \frac{\Delta^2}{3} \frac{1}{N_S} \sum_{\mathbf{k} \neq 0} \frac{n_0}{(Un_0 + A_{\mathbf{k}})^2}. \quad (16)$$

From Eq. (5), we see that the chemical potential μ is included in the expression of $A_{\mathbf{k}}$. Hence, in order to go further to discuss the behavior of the particle density, the chemical potential needs to be determined first. By minimizing the grand canonical potential $\bar{\Omega}$ in Eq. (12) with respect to the condensate density n_0 , i.e.

$(1/N_S) \partial \bar{\Omega} / \partial n_0 = 0$, we have the chemical potential to the lowest order [31]

$$\mu = Un_0 - Zt, \quad (17)$$

which makes the Bogoliubov spectrum (9) gapless in long-wavelength limit $\mathbf{k} \rightarrow \mathbf{0}$, accordant with the Nambu-Goldstone theorem [39, 40]. After taking the continuum limit [31], we finally obtain the 2D particle density expression of disordered lattice systems at zero-temperature

$$n = n_0 + n_I + n_R, \quad (18)$$

$$n_I = \frac{a^2}{2\pi^2} \int_0^{\pi/a} dk_1 \int_0^{\pi/a} dk_2 \frac{t_k}{\sqrt{t_k(2Un_0 + t_k)}} - \frac{1}{2}, \quad (19)$$

$$n_R = \frac{a^2}{3\pi^2} \Delta^2 \int_0^{\pi/a} dk_1 \int_0^{\pi/a} dk_2 \frac{n_0}{(2Un_0 + t_k)^2}, \quad (20)$$

where we have introduced $t_k = 4t - 2t \cos ak_1 - 2t \cos ak_2$ for simplicity.

Note that when disorder is set to be zero ($\Delta = 0$), our result exactly reduces to that obtained in clean lattice systems [31]. Recently, the finite temperature effect in Eq. (15) in clean cases has also been discussed [35]. In the following, we will focus on the disorder effect on the condensate density in the zero temperature limit.

IV. PHASE DIAGRAMS

To obtain the information of condensate fraction n_0/n from Eqs. (18), (19), and (20), we have two ways to follow. The first one is that along the commonly adopted way [29, 30], we perform a further approximation $n_0 \simeq n$ in the expressions of n_I in Eq. (19) and n_R in Eq. (20). With the help of Eq. (18), we can easily get the dependence of condensate fraction on disorder strength for different on-site interactions. As shown in Fig. 1 (Left), we find that our analytical result under the above mentioned approximation is in a good agreement with the previous numerical calculation [29] under the same approximation. The same result has also been produced in the framework of BT yet by a different technique [30].

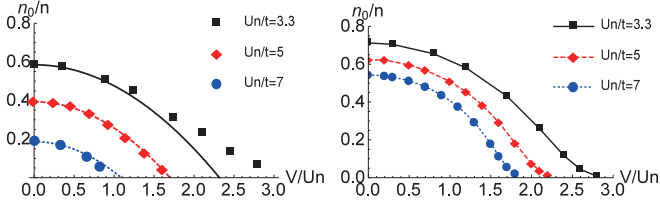


FIG. 1: (Color online) Condensate fraction n_0/n versus disorder strength V/Un for different interaction strengths Un/t ($V = \Delta/\sqrt{3}$, $n \simeq 0.33$ according to Refs. [29, 30]). Left: Points are from numerical results [29]; Lines come from our analytical result when taking $n_0 \simeq n$ in Eqs. (19) and (20); Right: Curves obtained by directly solving the implicit equations (18), (19), and (20).

The other way, which should be more accurate, is to solve directly the implicit equations Eqs. (18), (19), and (20). As is shown in Fig. 1 (right), this approach provides us different curves when compared with the left figure in Fig. 1. It is obvious that the approximation $n_0 \simeq n$ overestimates the condensate depletion. Hence, to reveal the behavior of the disordered system in weak interaction regime at zero temperature, the relative simple approximation of $n_0 \simeq n$ would not be adopted in the following.

By treating Eqs. (18), (19) and (20) as a set of equations, the dependence of the condensate fraction n_0/n on disorder strength Δ/t for different interaction strengths U/t can be determined rigorously via implicit function technique, see Fig. 2. From the picture, we see that, in the absence of disorder, n_0/n is less than unity because mutual interaction will scatter the particles out of the condensate state, and the scattering effect is intenser for stronger interaction strength, i.e. the condensate fraction is smaller for larger U/t . When disorder is introduced, the condensate fraction starts to decrease gradually. With the increase of disorder strength, n_0/n keeps going down at a relatively high speed. Remarkably, when disorder becomes sufficiently large, the decreasing of n_0/n dramatically slows down, leading to the long tail structures shown in Fig. 2.

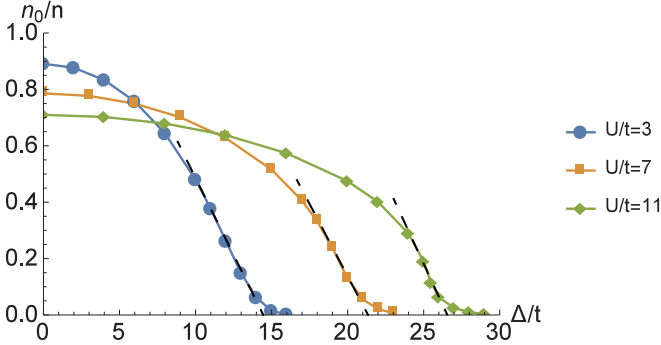


FIG. 2: (Color online) Condensate fraction n_0/n versus disorder strength Δ/t for different interaction strengths U/t . Dashed lines show the linear fits for fast decreasing parts of the curves, and are used to determine the transition points.

In addition, for different values of interaction strength, Fig. 2 shows cross behavior in the intermediate disorder region, indicating that the systems becomes more robust to the addition of disorder with the increase of U/t . Such a cross behavior has also been found in disordered systems without lattice potential [25].

It should be pointed out that in Fig. 2, n_0/n will never go rigorously down to zero at finite Δ , and this can also be easily read from the analytical expression Eqs. (18), (19), and (20), showing that MI will never be reached in weakly interacting regime. Although condensate fraction n_0/n is nonzero in both SF and BG phases [41], the rate of change of n_0/n is another topic. As displayed in Fig. 2, our calculation shows that the decreasing rate of n_0/n dramatically slows down when disorder is at large enough values. Such drastic decreasing of the change rate of n_0/n may be looked upon as a signal of the entrance of the system into BG phase from SF phase. The corresponding critical points Δ_c/t for different values of the on-site interaction can be approximately obtained by looking for the intersection points of the dashed lines and the asymptotes of the long tails of the curves (which are the horizontal axis) in Fig. 2. Such a procedure leads to the phase diagram shown in Fig. 3. Our result agrees qualitatively with the empirical square-root law $\Delta/t \propto \sqrt{U/t}$ [19, 21].

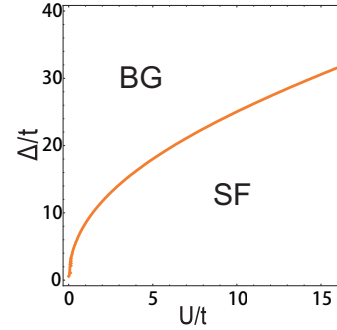


FIG. 3: (Color online) Zero temperature phase diagram of BG-SF phase transition in Δ/t - U/t plane obtained analytically by the use of Bogoliubov theory in the weakly interacting regime of 2D disordered BHM.

Thanks to the time-of-flight technique, the dependence of condensate depletion on optical lattice depth for different disorder strength can be directly measured in experiments [13, 36]. Meanwhile, it has been reported recently that Meldgin *et.al* have successfully probed the SF-BG transition of 3D disordered BHM by means of quantum quenches of disorder [14] and the corresponding disorder strength – lattice depth phase diagram has been obtained. Thus it would be helpful to find out the dependence of condensate fraction on lattice depth for different disorder strengths as well as the corresponding phase diagram in disorder – lattice depth plane for 2D systems.

In experiments, the preparation of a Bose gas in a thin

layer of 2D square optical lattice can be achieved by the following way [42, 43]. On the horizontal plane, two pairs of orthogonally crossed laser beams form a 2D square optical lattice. Atoms are confined vertically by introducing an additional vertical lattice, which is formed by two laser beams intersecting at a small angle relative to the horizontal plane. With a relative large site spacing, the vertical lattice helps the system form an array of 2D pancake potentials. When the strength of the vertical confinement is large enough, the tunneling in the vertical direction is effectively suppressed and becomes negligible.

As is known, under a single-band tight binding approximation, the Bose-Hubbard Hamiltonian (1) can be derived from a disordered many-body Hamiltonian [15]. The 2D on-site interaction strength can be expressed in terms of experimental parameters as [4]

$$\frac{U}{E_R} = \sqrt{\frac{4\pi V_0}{E_R}} \left(\frac{\hbar\omega_\perp}{E_R} \right) \frac{a_s}{a}, \quad (21)$$

and the asymptotic expression of hopping amplitude takes the form [44]

$$\frac{t}{E_R} = \frac{4}{\sqrt{\pi}} \left(\frac{V_0}{E_R} \right)^{3/4} \exp \left(-2\sqrt{\frac{V_0}{E_R}} \right), \quad (22)$$

where V_0 represents the optical lattice depth in horizontal plane and $a = \lambda/2$ is the lattice constant with λ being the wavelength of the laser. ω_\perp is the confinement frequency in third dimension. a_s is the s -wave scattering length and $E_R = 2\hbar^2\pi^2/m\lambda^2$ is the recoil energy. To plot the figure, we take the experimental parameters of a 2D optical lattice systems from Ref. [43], where a Bose gas of ^{133}Cs atoms is loaded and $\lambda = 1064\text{nm}$, $\omega_\perp = 1970 \cdot 2\pi\text{Hz}$, $a_s = 200a_B$ (a_B being the Bohr radius).

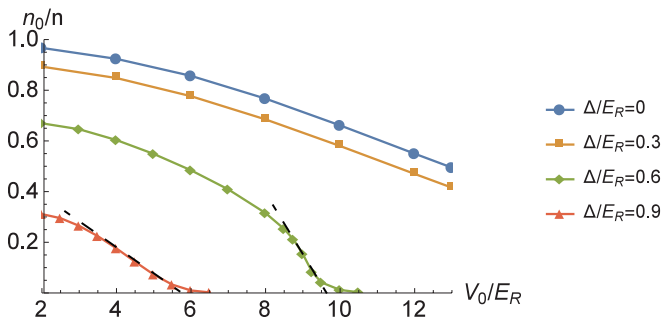


FIG. 4: (Color online) Condensate fraction n_0/n versus optical lattice strength V_0/E_R for different disorder strength Δ/E_R . Dashed lines show the linear fits used to determine the transition points.

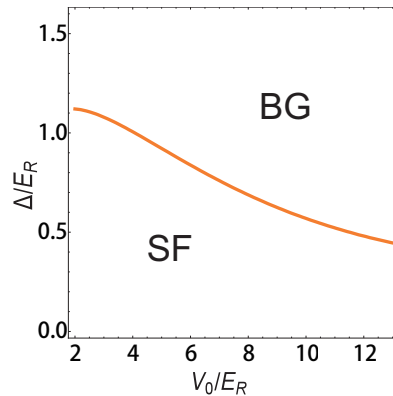


FIG. 5: (Color online) Zero temperature phase diagram of BG-SF obtained by Bogoliubov theory in the weakly interacting regime of 2D disordered BHM in terms of experimental parameters.

Combining the particle density expressions Eqs. (18), (19), and (20) with Eqs. (21), (22), we show the relationship between condensate fraction n_0/n and optical lattice depth V_0/E_R in Fig. 4. From this figure we see that although condensate fraction always remains at some finite value for weak disorder strength (where the on-site interaction is still dominant), it will approach to zero when the strength of disorder is strong enough. Similar behavior has been observed in 3D disordered optical lattices as well [13]. Thus the corresponding experiments in 2D and followed comparisons are expected.

Along the same avenue of obtaining Fig. 3, based on the result shown in Fig. 4, the $\Delta/E_R - V_0/E_R$ phase diagram of SF-BG phase transition for the 2D system is also determined, as shown in Fig. 5. We find that the critical disorder strength Δ_c/E_R decreases when lattice depth V_0/E_R goes high, which is in accordance with the experiment observations in 3D case [14]. Our result shown in Fig. 5 is expected to serve as a reference object for further studies, both theoretical and experimental, of 2D disordered Bose systems in optical lattices.

V. SUMMERY

In this paper we have applied Bogoliubov theory to dilute weakly interacting Bose gases in 2D optical lattices in the presence of uniformly distributed and uncorrelated disorder. Under the Bogoliubov approximation, we obtained the analytical expression of particle density in disordered lattice systems. While setting disorder strength to be zero, we found that our result exactly goes back to the former calculation gotten in the clean case [31]. By analyzing the implicit expressions of condensate fraction n_0/n , we obtained the relationships between n_0/n and disorder strength as well as lattice depth. Noticeably, a dramatic change of the decreasing rate of the condensate fraction in weakly interacting regime was observed. Associating such a distinct slowing down behavior to signal

the phase transition from SF to BG, the zero temperature phase boundary of SF-BG was determined. Our result turns out to be qualitatively in agreement with the empirical square-root law [19, 21] and could be examined experimentally in the near future.

Acknowledgments

The authors greatly acknowledge Axel Pelster for stimulating and fruitful discussions. This Work was sup-

ported by National Natural Science Foundation of China under Grant No. 11275119 and by Ph.D. Programs Foundation of Ministry of Education of China under Grant No. 20123108110004.

-
- [1] M. Lewenstein, A. Sanpera, V. Ahunger, B. Damski, A. Sen(De) and U. Sen, *Adv. Phys.* **56**, 243 (2007).
 - [2] O. Dutta, M. Gajda, P. Hauke, M. Lewenstein, D. S. Lühmann, B. A. Malomed, T. Sowiski and J. Zakrzewski, *Rep. Prog. Phys.* **78**, 066001(2015).
 - [3] I. Bloch, J. Dalibard and W. Zwerger, *Rev. Mod. Phys.* **80**, 885 (2008).
 - [4] K. V. Krutitsky, arXiv: 1501. 03125v1.
 - [5] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch and I. Bloch, *Nature* **415**, 39 (2002).
 - [6] M. P. A. Fisher, P. B. Weichman, G. Grinstein and D. S. Fisher, *Phys. Rev. B* **40**, 546 (1989).
 - [7] J. E. Lye, L. Fallani, M. Modugno, D. S. Wiersma, C. Fort and M. Inguscio, *Phys. Rev. Lett.* **95**, 070401 (2005).
 - [8] M. White, M. Pasienski, D. McKay, S. Q. Zhou, D. Ceperley, and B. DeMarco, *Phys. Rev. Lett.* **102**, 055301 (2009).
 - [9] F. Jendrzejewski, A. Bernard, K. Müller, P. Cheinet, V. Josse, M. Piraud, L. Pezze, L. Sanchez-Palencia, A. Aspect and P. Bouyer, *Nat. Phys.* **8**, 398 (2012).
 - [10] L. Fallani, J. E. Lye, V. Guarnera, C. Fort and M. Inguscio, *Phys. Rev. Lett.* **98**, 130404 (2007).
 - [11] B. Gadway, D. Pertot, J. Reeves, M. Vogt and D. Schneble, *Phys. Rev. Lett.* **107**, 145306 (2011).
 - [12] C. D'Errico, E. Lucioni, L. Tanzi, L. Gori, G. Roux, I. P. McCulloch, T. Giamarchi, M. Inguscio and G. Modugno, *Phys. Rev. Lett.* **113**, 095301 (2014).
 - [13] M. Pasienski, D. McKay, M. White and B. DeMarco, *Nat. Phys.* **6**, 677 (2010).
 - [14] C. Meldgin, U. Ray, P. Russ, D. Ceperley and B. DeMarco, arXiv: 1502. 02333v1.
 - [15] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner and P. Zoller, *Phys. Rev. Lett.* **81**, 3108 (1998).
 - [16] A. E. Niederle and H. Rieger, *New J. Phys.* **15**, 075029 (2013).
 - [17] L. Pollet, N. V. Prokofev, B. V. Svistunov and M. Troyer, *Phys. Rev. Lett.* **103**, 140402 (2009).
 - [18] V. Gurarie, L. Pollet, N. V. Prokofev, B. V. Svistunov and M. Troyer, *Phys. Rev. B* **80**, 214519 (2009).
 - [19] S. G. Söyler, M. Kiselev, N. V. Prokofev and B. V. Svistunov, *Phys. Rev. Lett.* **107**, 185301 (2011).
 - [20] C. Zhang, A. Safavi-Naini and B. Capogrosso-Sansone, *Phys. Rev. A* **91**, 031604(R) (2015).
 - [21] G. M. Falco, T. Nattermann and V. L. Pokrovsky, *Europhys. Lett.* **85**, 30002 (2009); *Phys. Rev. B* **80**, 104515 (2009).
 - [22] N. N. Bogoliubov, *Izv. Acad. Nauk (USSR)* **11**, 77 (1947) [*J. Phys.* **11**, 23 (1947)].
 - [23] K. Huang and H. F. Meng, *Phys. Rev. Lett.* **69**, 644 (1992).
 - [24] G. E. Astrakharchik, J. Boronat, J. Casulleras and S. Giorgini, *Phys. Rev. A* **66**, 023603 (2002).
 - [25] G. E. Astrakharchik, K. V. Krutitsky and P. Navez, *Phys. Rev. A* **87**, 061601(R) (2013).
 - [26] G. M. Falco, A. Pelster and R. Graham, *Phys. Rev. A* **75**, 063619 (2007); M. Ghabour and A. Pelster, *Phys. Rev. A* **90**, 063636 (2014).
 - [27] C. Gaul, A. Müller, *Appl. Phys. B* **117**, 775C784 (2014).
 - [28] J. Saliba, P. Lugan and V. Savona, *Phys. Rev. A* **90**, 031603(R) (2014).
 - [29] K. G. Singh and D. S. Rokhsar, *Phys. Rev. B* **49**, 9013 (1994).
 - [30] C. Gaul and C. A. Müller, *Eur. Phys. J. Special Topics* **217**, 69 (2013).
 - [31] D. van Oosten, P. van der Straten and H. T. C. Stoof, *Phys. Rev. A* **63**, 053601 (2001).
 - [32] R. Roth and K. Burnett, *Phys. Rev. A* **67**, 031602(R) (2003).
 - [33] G. Orso, C. Menotti and S. Stringari, *Phys. Rev. Lett.* **97**, 190408 (2006).
 - [34] V. I. Yukalov, *Condens. Matter Phys.* **16**, 23002 (2013).
 - [35] M. O. C. Pires and E. J. V. de Passos, arXiv: 1505. 01875v1.
 - [36] K. Xu, Y. Liu, D. E. Miller, J. K. Chin, W. Setiawan and W. Ketterle, *Phys. Rev. Lett.* **96**, 180405 (2006).
 - [37] P. T. Ernst1, S. Götze1, J. S. Krauser1, K. Pyka1, D. S. Lühmann, D. Pfannkuche and K. Sengstock, *Nat. Phys.* **6**, 56 (2010).
 - [38] U. Bissbort, S. Götze, Y. Li, J. Heinze, J. S. Krauser, M. Weinberg, C. Becker, K. Sengstock and W. Hofstetter, *Phys. Rev. Lett.* **106**, 205303 (2011).
 - [39] Y. Nambu, *Phys. Rev. Lett.* **4**, 380 (1960).
 - [40] J. Goldstone, *Nuovo Cimento* **19**, 154 (1961).
 - [41] P. Buonsante, V. Penna, A. Vezzani and P. B. Blakie, *Phys. Rev. A* **76**, 011602(R) (2007).
 - [42] N. Gemelke, X. Zhang, C.-L. Hung and C. Chin, *Nature* **460**, 995 (2009).
 - [43] C. L. Hung, X. Zhang, N. Gemelke and C. Chin, *Phys. Rev. Lett.* **104**, 160403 (2010).
 - [44] W. Zwerger, *J. of Opt. B: Quantum and Semiclass. Opt.* **5**: S9, 2003.